

TIME-DELAY OPERATORS IN SEMICLASSICAL LIMIT. II. SHORT-RANGE POTENTIALS

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ABSTRACT. This work is a continuation of [27]. We prove that quantum time-delay operator localized in a nontrapping energy interval is in fact an h -pseudo-differential operator with the h -principal symbol given by the classical time-delay function composed with the incoming wave operator in classical mechanics. The classical limit of time-delay operator is also given.

1. INTRODUCTION

Let $H_0^h = -h^2\Delta$, $H^h = H_0^h + V(x)$, where $h \in]0, h_0]$ is a small parameter and V is a real, smooth function on \mathbf{R}^n satisfying, for some $\varepsilon_0 > 0$,

$$(1.1) \quad |\partial_x^\alpha V(x)| \leq c_\alpha \langle x \rangle^{-1-|\alpha|-\varepsilon_0}, \quad x \in \mathbf{R}^n, \quad \alpha \in \mathbf{N}^n.$$

Here $\langle x \rangle = (1 + |x|^2)^{1/2}$. Put $U_0(t, h) = \exp(ih^{-1}tH_0^h)$ and $U(t, h) = \exp(-ih^{-1}tH^h)$. Define the wave operators $W_\pm(h)$ by

$$W_\pm(h) = \text{s-lim}_{t \rightarrow \pm\infty} U(t, h)^* U_0(t, h), \quad \text{in } L^2(\mathbf{R}^n).$$

Let $S(h) = W_+(h)^* W_-(h)$ be the scattering operator. Then it is known (cf. [1, 2, 13, 23–24], and references cited therein) that the time-delay operator for the scattering process (H_0^h, H^h) , defined by taking the large space limit of the difference between sojourn times of the free and interacting particles, exists and is given by:

$$(1.2) \quad H_0^h T(h) = \frac{1}{2} S(h)^* [S(h), A(h)], \quad A(h) = \frac{h}{2} (x \cdot D_x + D_x \cdot x).$$

Equation (1.2) makes sense on the set $D = \{f \in L^2(\mathbf{R}^n); \hat{f} \in C_0(\mathbf{R}^n \setminus \{0\})\}$. In classical scattering theory, there is a time-delay function defined via sojourn times of classical particles (see [11]). Consider the Hamilton system:

$$(1.3) \quad \begin{cases} \dot{q}(t) = 2p(t), & q(0) = x, \\ \dot{p}(t) = -\nabla V(q(t)), & p(0) = \xi. \end{cases}$$

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If $(q(t; x, \xi), p(t; x, \xi))$ is a scattering trajectory, i.e., $\lim_{t \rightarrow \pm\infty} |q(t; x, \xi)| = +\infty$, then the classical time-delay function τ can be defined at (x, ξ) and is equal to:

$$(1.4) \quad \tau(x, \xi) = \frac{1}{2E} \left\{ \lim_{t \rightarrow -\infty} (q(t) - 2tp(t)) \cdot p(t) - \lim_{t \rightarrow +\infty} (q(t) - 2tp(t)) \cdot p(t) \right\},$$

$E = |\xi|^2 + V(x)$. In this paper, we use the following nontrapping assumption to assure τ to be defined. Let $J \subset]0, +\infty[$ be an open interval. We shall call J an interval of nontrapping energy, if for every subinterval $I \subset \subset J$, and for every $R > 0$, there exists $t_0 > 0$ such that

$$(1.5) \quad |q(t; x, \xi)| > R, \quad \text{for } |t| \geq t_0$$

and for $(x, \xi) \in p^{-1}(I)$ with $|x| < R$. Here $p(x, \xi) = |\xi|^2 + V(x)$. From our earlier works, it is clear that the nontrapping condition is important in studying semiclassical propagation properties of scattering states for H^h . See [25, 26].

Before stating the main results of this work, let us introduce some notations. For $\varepsilon \in]0, 2[$, $d > 0$, $R > 0$, we define:

$$\Omega_{\pm}(\varepsilon, d, R) = \{(x, \xi) \in \mathbf{R}^{2n}; \pm \widehat{x} \cdot \widehat{\xi} > -1 + \varepsilon, |\xi|^2 > d, |x| > R\}.$$

Here $\widehat{x} = x/|x|$ and $\widehat{\xi} = \xi/|\xi|$. Put $\Omega(\varepsilon, d, R) = \Omega_{+}(\varepsilon, d, R) \cup \Omega_{-}(\varepsilon, d, R)$. For $m \in \mathbf{R}$, introduce the class of symbols on \mathbf{R}^{2n} :

$$S^m = \{a \in C^{\infty}(\mathbf{R}^{2n}); |\partial_x^{\alpha} \partial_{\xi}^{\beta} a(x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{m-|\alpha|}\}.$$

Put $S_{\pm}^m(\varepsilon, d, R) = \{a \in S^m; \text{supp } a \subset \Omega_{\pm}(\varepsilon, d, R)\}$ and $S_{\pm}^m = \bigcup S_{\pm}^m(\varepsilon, d, R)$, where the union is taken over $\varepsilon \in]0, 2[$, $d > 0$, $R > 0$. If $a(h) \in C^{\infty}(\mathbf{R}^{2n})$ has an asymptotic expansion of the form:

$$a(h) \sim \sum_{j=0}^{\infty} h^{N+j} a_j, \quad a_j \in S^{m-j},$$

we shall just write: $a(h) \in S^{m, N}$. Similarly, we can define the elements for $S_{\pm}^{m, N}$. For a symbol $a(h)$, we define an h -pseudodifferential operator (h - ψ DO) by

$$(1.6) \quad a(x, hD; h)u(x) = (2\pi h)^{-n} \iint e^{i(x-y) \cdot \xi/h} a(x, \xi; h)u(y) dy d\xi,$$

$$u \in \mathcal{S}(\mathbf{R}^n).$$

In the following, we shall not distinguish an h - ψ DO from an h -admissible operator in the sense of Helffer and Robert [4], which is, by definition, an h - ψ DO modulo an error of order $O(h^{\infty})$ in $\mathcal{L}(L^2)$. The main results of this paper are the following.

Theorem 1. Assume that V satisfies (1.1) and J is an interval of nontrapping energy. Let $a, b \in C_0^\infty(\mathbf{R}_x^n)$ or $a, b \in S_+^0 \cap S_-^0$. In the former case, we shall write: $a(x, \xi) = a(x)$, $\forall \xi \in \mathbf{R}^n$, etc. Let $f \in C_0^\infty(J)$. Then $a(x, hD)f(H_0^h)T(h)b(x, hD)$ is an h - ψ DO of order zero. More precisely, there exists $t(h) \in S^{0,0}$, $t(h) \sim \sum_{j=0}^\infty h^j t_j$, with

$$\|a(x, hD)f(H_0^h)T(h)b(x, hD) - t(x, hD; h)\|_{\mathcal{L}(L^2)} = O(h^\infty), \quad h \rightarrow 0_+.$$

Moreover, $t_0(x, \xi) = a(x, \xi)b(x, \xi)f(|\xi|^2)\tau \circ \Omega_-^{\text{cl}}(x, \xi)$. Here τ is the classical time-delay function and Ω_-^{cl} is the incoming wave operator in classical mechanics defined by

$$\Omega_-^{\text{cl}}(x, \xi) = \lim_{t \rightarrow +\infty} \phi^{-t} \circ \phi_0^t(x, \xi), \quad \xi \neq 0.$$

Here $\phi^t = (q(t), p(t)) = (x + 2t\xi, \xi)$ is the solution of (1.3) and $\phi_0^t(x, \xi)$.

From Theorem 1, we can easily derive the classical limit of $T(h)$ applied to coherent states. For this purpose, it is natural to regard x and ξ as symmetric variables. Thus introduce U_h by

$$U_h f(x) = h^{-n/4} f(h^{-1/2}x), \quad f \in L^2(\mathbf{R}^n).$$

Put: $\overline{H}_0^h = U_h^* H_0^h U_h$, $\overline{H}^h = U_h^* H^h U_h$. Then the time-delay operator corresponding to $(\overline{H}_0^h, \overline{H}^h)$ is given by

$$(1.8) \quad \overline{T}(h) = U_h^* T(h) U_h.$$

Theorem 2. Under the assumptions of Theorem 1, let $(x_0, \xi_0) \in \mathbf{R}^{2n}$, with $|\xi_0|^2 \in J$, and $f \in C_0^\infty(J)$ with $f(|\xi_0|^2) = 1$. Denote

$$W_h(x_0, \xi_0) = \exp(ih^{-1/2}(x \cdot \xi_0 - x_0 \cdot D_x)).$$

Then one has

$$(1.9) \quad \lim_{h \rightarrow 0_+} \langle W_h(x_0, \xi_0)^* f(\overline{H}_0^h) \overline{T}(h) W_h(x_0, \xi_0) f, g \rangle = \tau \circ \Omega_-^{\text{cl}}(x_0, \xi_0) \langle f, g \rangle$$

for any $f, g \in L^2(\mathbf{R}^n)$.

Theorems 1 and 2 illustrate clearly the relation between quantum and classical time-delay. Remark that semiclassical approximation of scattering quantities was studied by several authors. See for example [18–20, 27, 28]. In particular, in [18], Robert and Tamura considered the asymptotic of trace of on-shell time-delay operator $T(\lambda, h) \in \mathcal{L}(L^2(S^{n-1}))$ under the assumption $V(x) = O(|x|^{-n-\varepsilon_0})$. In [27], I proved Theorems 1 and 2 for $V \in C_0^\infty(\mathbf{R}^n)$. The method used there is based on the correspondence between quantum and classical dynamics [21, 22]. By that method, the leading term of $T(h)$ in semiclassical approximation is easy to find. But serious difficulties arise in estimating remainders. In this paper, our proof for Theorems 1 and 2 relies heavily on

energy decay estimates given in §2. These estimates enable us to obtain that there exists $T_0 > 0$ such that, with suitable microlocalization, one has:

$$(1.10) \quad W_{\pm}(h) = U(t, h)^* J^{\pm} U_0(t, h) = O(h^{\infty}), \quad \text{in } \mathcal{L}(L^2)$$

for $\pm t > T_0$. Here J^{\pm} is outgoing (or incoming) h -parametrix constructed with a phase slightly different from that of Isozaki and Kitada [6]. Notice that the method of this paper can also be used to study the semiclassical approximation of wave operators. In this case the nontrapping condition is not needed, for the classical wave operators are well defined for $(x, \xi) \in R^{2n}$ with $\xi \neq 0$. See [16].

The organization of this paper is as follows. In §2, we give microlocalized energy decay estimates for $U(t, h)$. The difference between Proposition 2.1 and my previous works [25, 26] is that under suitable microlocalizations, we can obtain estimates of the form: $O(h^N \langle t \rangle^{-N})$, for $\pm t > 0$ and for any $N > 0$. These estimates are crucial to the semiclassical approximation of $T(h)$. §3 is devoted to classical dynamics. In §4, we give proof for Theorems 1 and 2. Finally we recall in the Appendix a semiclassical Egorov theorem, which is used in §4.

2. MICROLOCALIZED TIME-DECAY ESTIMATES

In this section, V is assumed to be a long-range potential satisfying, for some $\varepsilon_0 > 0$,

$$(2.1) \quad |\partial_x^{\alpha} V(x)| \leq C_{\alpha} \langle x \rangle^{-\varepsilon_0 - |\alpha|}, \quad x \in \mathbf{R}^n, \quad \alpha \in \mathbf{N}^n.$$

We want to give some microlocalized time-decay estimates on $U(t, h)$. In the case of fixed $h > 0$, this type of question was considered by several authors (cf. [3, 5–9]). In the semiclassical case, the local geometrical form of the potential plays an important role. By method of incoming and outgoing parametrices of Isozaki and Kitada, I proved in [25, 26] that the nontrapping condition (1.4) is both necessary and sufficient to obtain optimal results of the form

$$\| \langle x \rangle^s b_{\pm}(x, hD) f(H^h) U(t, h) \langle x \rangle^{-s-r} \| = O(\langle t \rangle^{-r}), \quad \forall t > 0,$$

uniformly in $h \in]0, h_0]$, where $b_{\pm} \in S_{\pm}^0$, $f \in C_0^{\infty}(J)$ and $s, r \geq 0$. The following improvements are useful to semiclassical approximation of $T(h)$.

Proposition 2.1. *Assume that V satisfies (2.1). Let J be an interval of nontrapping energy. Let $f \in C_0^{\infty}(J)$, and $m \in \mathbf{R}$. Then we have:*

(i) *For any $\sigma, d, R_0 > 0$, there exists $R = R(\sigma, d, R_0) > 0$ such that for any $\chi \in C_0^{\infty}(B_{R_0})$, $b_{\pm} \in S_{\pm}^m(\sigma, d, R)$, one has*

$$(2.2) \quad \| \chi f(H^h) U(t, h) b_{\pm}(\chi, hD) \| \leq C_N h^N \langle t \rangle^{-N}, \quad \pm t > 0, \quad 0 < h \leq h_0,$$

for any $N \in \mathbf{N}$.

(ii) Assume that $b_{\pm} \in S_{\pm}^m(\sigma_{\pm}, d_{\pm}, R_{\pm})$ with $\sigma_+ + \sigma_- > 2$ and $R_+ + R_-$ large enough. Then for any $N \in \mathbf{N}$, one has

$$(2.3) \quad \|b_{\pm}(x, hD)f(H^h)U(t, h)b_{\mp}(x, hD)\| \leq C_N h^N \langle t \rangle^{-N}$$

for $\mp t > 0$, $h \in]0, h_0]$.

Proof. It can be easily checked that (2.2) and (2.3) are true if we replace H^h by H_0^h and $U(t, h)$ by $U_0(t, h)$. In the general case, we follow the line of [25, 26]. By constructing outgoing and incoming h -parametrixes as in [25], making use of Theorem 1 in [26], we can compare directly $U(t, h)$ with $U_0(t, h)$ and derive the desired results by the arguments already used in [25]. The details are omitted. \square

Corollary 2.2. Let $S_{\pm}^m = \bigcup S_{\pm}^m(\sigma, d, R)$, where the union is taken over $\sigma \in]0, 2[$, $d > 0$, $R > 0$. For any $\chi_1, \chi_2 \in C_0^{\infty}(\mathbf{R}_x^n)$, $b_{\pm} \in S_{\pm}^m$, there exists some $T_0 > 0$ such that

$$(2.4) \quad \|\chi_1 f(H^h)U(t, h)\chi_2\| = O(h^{\infty} \langle t \rangle^{-\infty}), \quad |t| > T_0,$$

$$(2.5) \quad \|\chi_1 f(H^h)U(t, h)b_{\pm}(x, hD)\| = O(h^{\infty} \langle t \rangle^{-\infty}), \quad \pm t > T_0.$$

If $B_{\pm} \in S_{\pm}^m(\sigma_{\pm}, d_{\pm}, R_{\pm})$ with $\sigma_+ + \sigma_- > 2$, then there exists $T_0 > 0$ such that

$$(2.6) \quad \|b_{\mp}(x, hD)f(H^h)U(t, h)b_{\pm}(x, hD)\| = O(h^{\infty} \langle t \rangle^{-\infty}), \quad \pm t > T_0.$$

Here $f \in C_0^{\infty}(J)$ and J is an interval of nontrapping energy.

Proof. Take $g \in C_0^{\infty}(J)$ such that $g = 1$ on $\text{supp } f$. Then we can write

$$g(H^h)\chi_2(x) = b(x, hD; h) + R(h)$$

where $\|\langle x \rangle^N R(h)\langle x \rangle^N\| = O(h^N)$ for any $N > 0$ and $b(h)$ is a symbol with support in $\{(x, \xi); x \in \text{supp } \chi_2, p(x, \xi) \in \text{supp } g\}$. By a semiclassical Egorov theorem (see the Appendix), one has

$$U(T, h)b(x, hD; h) = b_T(x, hD; h)U(T, h) + R_T(h), \quad T > 0,$$

where $R_T(h)$ has the same properties as $R(h)$ and

$$(2.7) \quad \text{supp } b_T(h) \subset \text{supp } b(\phi^{-T}, h).$$

Here ϕ^t is the Hamilton flow for $p = |\xi|^2 + V(x)$. Therefore if $(x, \xi) \in \text{supp } b_T(h)$, we can write: $(x, \xi) = \phi^T(y, \eta)$ for some $(y, \eta) \in \text{supp } b(h)$. Since $|y| \leq R_0$ for some $R_0 > 0$ independent of T , the nontrapping assumption implies that for any $R > 0$ there exists $T_0 > 0$ such that for $T > T_0$, $\text{supp } b_T(h) \subset \Omega_+(1, d_0, R)$. If $R > 0$ is large enough, we can apply (2.2) to obtain that

$$(2.8) \quad \|\chi_1 f(H^h)U(t, h)b(x, hD; h)\| = O(h^{\infty} \langle t \rangle^{-\infty}), \quad t > T_0.$$

From Theorem 1 in [26] and (2.8), one derives (2.4) for $t > T_0$. The case $-t > T_0$ can be obtained by taking the adjoint. Equations (2.5) and (2.6) can be proved in the same way. It should be noticed that if I is a compact subinterval contained in J , then there exist $C > 0$ and $t_0 > 0$ such that $|q(t; x, \xi)| \geq C(|x| + t|\xi|)$, for $t > t_0$, $(x, \xi) \in \text{supp } b_+ \cap p^{-1}(I)$. In particular, one can derive that for any $\varepsilon > 0$ and $R > 0$, one can take $T > 0$ large enough such that

$$\phi^T(\text{supp } b_+ \cap p^{-1}(I)) \subset \Omega_+(2 - \varepsilon, d, R).$$

This enables us to obtain (2.5) and (2.6) from Proposition 2.1. The details are omitted. \square

3. CLASSICAL DYNAMICS

From now on, the potential V is always assumed to satisfy (1.1). In this section, we want to establish some results on classical dynamics, which are useful in the next section. Our first result is concerned with an improvement of a result of Isozaki and Kitada [6, Proposition 2.4]. This improvement is of course due to the short range assumption on V , but it does not follow directly from the construction given in [6].

Lemma 3.1. *Let V satisfy (1.1). Then for any $\varepsilon, d > 0$, there exist two real functions $\phi_{\pm} \in C^\infty(\mathbf{R}^{2n})$ such that for R large enough, ϕ_{\pm} satisfies the eikonal equation*

$$(3.1) \quad |\nabla_x \phi_{\pm}(x, \xi)|^2 + V(x) = |\xi|^2 \quad \text{in } \Omega_{\pm}(\varepsilon, d, R_0^*)$$

and for any multi-indices $\alpha, \beta \in \mathbf{N}^n$,

$$(3.2) \quad |\partial_x^\alpha \partial_\xi^\beta (\phi_{\pm}(x, \xi) - x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{-\varepsilon_0 - |\alpha|}, \quad (x, \xi) \in R^{2n}.$$

Proof. Let $\phi^t(x, \xi) = (q(t; x, \xi), p(t; x, \xi))$ denote the solution of Hamilton system (1.3). Then we can verify that for $0 < \varepsilon < 2d > 0$, there exist some $C > 0$ and $R_0 > 0$ such that for $R > R_0$, $(x, \xi) \in \Omega_+(\varepsilon, d, R)$, we have

$$(3.3) \quad \phi^t(x, \xi) \in \Omega_+(\varepsilon/2, d/2, R/C) \quad \text{and} \quad |q(t; x, \xi)| \geq C^{-1}(|x| + t|\xi|)$$

for all $t > 0$. Using the expression

$$p(t; x, \xi) = \xi - \int_0^t \nabla V(q(s; x, \xi)) ds,$$

we can easily derive that

$$|p(t; x, \xi) - \xi| \leq C \langle x \rangle^{-1 - \varepsilon_0}, \quad \text{for all } t > 0.$$

For $|x| > R$, define $\Gamma_+(\varepsilon, d) = \{\xi \in R^n; (x, \xi) \in \Omega_+(\varepsilon, d, R)\}$. From the above estimates, it follows that for $R > 0$ large enough, and for fixed x with $|x| > R$, the map $\xi \rightarrow p(t; x, \xi)$ is a global diffeomorphism from $\Gamma_+(\varepsilon, d)$ onto its image, whose union, when x ranges over the domain $\{|y| > R\}$,

contains a set of the form $\Omega_+(2\varepsilon, 2d, C\mathbf{R})$ for all $t > 0$. Let $\xi \rightarrow \eta(t; x, \xi)$ be the inverse to $\xi \rightarrow p(t; x, \xi)$. Put

$$y(t; x, \xi) = q(t; x, \eta(t; x, \xi)).$$

Then one has:

$$\begin{aligned} y(t; x, \xi) &= x + 2t\xi + \int_0^t 2s \nabla V(q(s; x, \eta(s; x, \xi))) ds \\ &= x + 2t\xi + O(\langle x \rangle^{-\varepsilon_0}). \end{aligned}$$

From this estimate, one derives easily that $\nabla_x y(t; x, \xi) = I + O(\langle x \rangle^{-\varepsilon_0})$. With a little more work, we can in fact prove that

$$(3.4) \quad \nabla_x y(t; x, \xi) = I + O(\langle x \rangle^{-1-\varepsilon_0})$$

uniformly in $t > 0$. Finally we define as in [6]

$$\begin{aligned} u(t; x, \xi) &= x \cdot \xi + \int_0^t (p - x \cdot \nabla V)(\phi^s(x, \xi)) ds, \\ \phi(t; x, \xi) &= u(t; x, \eta(t; x, \xi)) \end{aligned}$$

for $(x, \xi) \in \Omega_+(\varepsilon, d, R_0)$, with $R_0 = R_0(\varepsilon, d)$ large enough. Then by the Hamilton-Jacobi theory,

$$(3.5) \quad \begin{cases} \frac{\partial}{\partial t} \phi(t; x, \xi) = |\xi|^2 + V(\nabla_\xi \phi(t; x, \xi)) = |\nabla_x \phi(t; x, \xi)|^2 + V(x), \\ \phi(0; x, \xi) = x \cdot \xi \quad \text{for } (x, \xi) \in \Omega_+(\varepsilon, d, R_0) \end{cases}$$

and

$$(3.6) \quad \partial_x \phi(t; x, \xi) = \eta(t; x, \xi); \quad \partial_\xi \phi(t; x, \xi) = y(t; x, \xi).$$

Now instead of taking ϕ_+ as the limit $\lim_{t \rightarrow +\infty} (\phi(t; x, \xi) - \phi(t; 0, \xi))$, we define it as

$$\phi_+(x, \xi) = \lim_{t \rightarrow +\infty} (\phi(t; x, \xi) - t|\xi|^2).$$

Then ϕ_+ satisfies the eikonal equation in $\Omega_+(\varepsilon, d, R_0)$ and

$$\phi_+(x, \xi) - x \cdot \xi = \int_0^{+\infty} V(y(t; x, \xi)) dt, \quad (x, \xi) \in \Omega_+(\varepsilon, d, R_0).$$

From (3.4) and similar estimates over higher derivatives, it follows that (3.2) is satisfied. Now to finish the proof of lemma for ϕ_+ , it suffices to introduce a suitable cut-off function to make ϕ_+ globally defined on \mathbf{R}^{2n} . ϕ_- can be constructed in a similar way. \square

Take $\chi \in C_0^\infty(\mathbf{R}^n)$, $\chi(x) = 0$ for $|x| \leq 1$, $\chi(x) = 1$ for $|x| \geq 2$. For $R_1 > R_0$, replacing ϕ_\pm by $(\phi_\pm(x, \xi) - x \cdot \xi)\chi(x/R_1) + x \cdot \xi$ in (3.1), one sees clearly that we can assume without loss that ϕ_\pm satisfies:

$$(3.7) \quad |\nabla_x \phi_\pm(x, \xi)|^2 + V(x) = |\xi|^2 \quad \text{in } \Omega_\pm(\varepsilon, d, R_1),$$

$$(3.8) \quad |\partial_x^\alpha \partial_\xi^\beta (\phi_\pm(x, \xi) - x \cdot \xi)| \leq c_{\alpha\beta} R_1^{-e_1} \langle x \rangle^{-|\alpha| - \varepsilon_2}, \quad (x, \xi) \in \mathbf{R}^{2n},$$

with $\varepsilon_1 + \varepsilon_2 = \varepsilon_0$, $\varepsilon_j \geq 0$, and $C_{\alpha\beta}$ independent of R_1 . For R_1 large enough, the maps $\xi \rightarrow \nabla_x \phi_{\pm}(x, \xi)$, $x \rightarrow \nabla_{\xi} \phi_{\pm}(x, \xi)$ are global diffeomorphisms. Let $\eta_{\pm}(x, \xi)$, $y_{\pm}(x, \xi)$ denote the inverse diffeomorphism:

$$(3.9) \quad \nabla_x \phi_{\pm}(x, \eta_{\pm}(x, \xi)) = \xi, \quad \nabla_{\xi} \phi_{\pm}(y_{\pm}(x, \xi), \xi) = x, \quad (x, \xi) \in R^{2n}.$$

Lemma 3.2. *Let $(q(t; x, \xi), p(t; x, \xi))$ denote the solution of the Hamilton system (1.3). Then for any $\sigma, d > 0$, there exists $R_1 > 0$ such that for $(x, \xi) \in \Omega_{\pm}(\sigma, d, R_1)$, one has*

$$(3.10) \quad q(t; x, \xi) = \nabla_{\xi} \phi_{\pm}(x, \eta_{\pm}(x, \xi)) + 2t\eta_{\pm}(x, \xi) + O(\langle t \rangle^{-\varepsilon_0}), \quad t \rightarrow \pm\infty.$$

In particular for $(x, \xi) \in \Omega_{\pm}(\sigma, d, R_1)$,

$$\begin{cases} \nabla_{\xi} \phi_{\pm}(x, \eta_{\pm}(x, \xi)) = x + \int_0^{\pm\infty} s \nabla V(q(s; x, \xi)) ds, \\ \eta_{\pm}(x, \xi) = \xi - \int_0^{\pm\infty} \nabla V(q(s; x, \xi)) ds. \end{cases}$$

Proof. Consider only the case $t \rightarrow +\infty$. Recall that for $(x, \xi) \in \Omega_+(\sigma, d, R_1)$ with R_1 large enough, one has

$$\phi_+(x, \xi) = \lim_{t \rightarrow +\infty} (\phi(t; x, \xi) - t\xi^2)$$

where $\phi(t; x, \xi)$ satisfies (3.5) and (3.6). It is easy to check that

$$\partial_{\xi} \phi(t; x, \xi) = \nabla_{\xi} \phi_+(x, \xi) + 2t\xi + o(1), \quad t \rightarrow +\infty.$$

Since for $(x, \xi) \in \Omega_+(\sigma, d, R_1)$, the limit $\eta(\infty, x, \xi) = \lim_{t \rightarrow +\infty} \eta(t; x, \xi)$ exists and

$$(3.11) \quad \begin{aligned} |\partial_x^{\alpha} \partial_{\xi}^{\beta} q(t; x, \xi)| &\leq C_{\alpha\beta} \langle t \rangle, \\ |\partial_x^{\alpha} \partial_{\xi}^{\beta} (\eta(t; x, \xi) - \eta(\infty, x, \xi))| &\leq C_{\alpha\beta} \langle t \rangle^{-1-\varepsilon_0}, \end{aligned}$$

one derives from the equality $\partial_{\xi} \phi(t; x, \xi) = q(t; x, \eta(t; x, \xi))$ that

$$(3.12) \quad q(t; x, \eta(\infty, x, \xi)) = \partial_{\xi} \phi_+(x, \xi) + 2t\xi + o(1), \quad t \rightarrow +\infty.$$

Noticing that $\eta(\infty; x, \xi) = \lim_{t \rightarrow \infty} \nabla_x \phi_+(t; x, \xi) = \nabla_x \phi_+(x, \xi)$, (3.10) for $t \rightarrow +\infty$ follows easily from (3.12) by returning to the Hamilton system. \square

Remark that for the phase functions constructed by Isozaki and Kitada, one has

$$q(t; x, \xi) = a_{\pm}(x, \xi) + 2t\eta_{\pm} + o(\langle t \rangle^{-\varepsilon_0}), \quad t \rightarrow \pm\infty,$$

where $a_{\pm}(x, \xi) = \nabla_{\xi} \phi_{\pm}(x, \eta_{\pm}(x, \xi)) + b_{\pm}(\xi)$, $b_{\pm} \neq 0$. This is clear from their definition of ϕ_{\pm} and from an estimate similar to (3.2). Now let σ, d , and R_1 be as in Lemma 3.2 and ϕ_{\pm} be solutions to the eikonal equation constructed in Lemma 3.1. Let $a_{0,\pm}$ denote the solution to the equation

$$(3.13) \quad 2\nabla_x \phi_{\pm} \cdot \nabla_x a_{0,\pm} + \Delta_x \phi_{\pm} a_{0,\pm} = 0, \quad \text{in } \Omega_{\pm}(2\sigma, 2d, 2R_1),$$

with the condition $a_{0,\pm}(x, \xi) = 1 + o(\langle x \rangle^{-\varepsilon_0})$, $(x, \xi) \in \Omega_{\pm}(2\sigma, 2d, 2R_1)$. Such solutions exist [25] and are in fact the principal symbol of the outgoing and incoming h -parametrixes used in §4.

Lemma 3.3. *Let $a_{0,\pm}$ be defined as above. Then for $(x, \xi) \in \Omega_{\pm}(2\sigma, 2d, 2R_1)$, one has*

$$a_{0,\pm}(x, \xi) = |\det(\partial_x \partial_{\xi} \phi_{\pm}(x, \xi))|^{1/2}.$$

Proof. Consider only $a_{0,+}$. As in [25], we can write

$$a_{0,+}(x, \xi) = e^{\frac{1}{2} \int_0^{+\infty} \Delta_x \phi_+(\rho(t; x, \xi), \xi) dt}$$

where $\rho(t)$ is the solution of the problem

$$\dot{\rho}(t) = \nabla_x \phi_+(\rho(t), \xi), \quad \rho(0; x, \xi) = x.$$

By the proof of Lemma 3.2, $q_0(t; x, \xi) \equiv q(t; x, \nabla_x \phi_+(x, \xi))$ satisfies

$$(3.14) \quad \begin{cases} q_0(0; x, \xi) = x, & \lim_{t \rightarrow +\infty} q_0(t; x, \xi) = 2\xi, \\ \partial_x q_0(t; x, \xi) = \partial_x \partial_{\xi} \phi_+(x, \xi) + O(\langle t \rangle^{-\epsilon_0}), & t \rightarrow +\infty. \end{cases}$$

In addition, by the Hamilton-Jacobi theory, we have:

$$\dot{q}_0(t; x, \xi) = \nabla_x \phi_+(q_0(t; x, \xi), \xi).$$

This means: $\rho(t; x, \xi) = q_0(t; x, \xi)$ for $t > 0$ and $(x, \xi) \in \Omega_+(2\sigma, 2d, 2R_1)$ with $R_1 > 0$ large enough. From Liouville theorem (see [12]), we obtain

$$\exp \left(\int_0^t \Delta_x \phi_+(q_0(s; x, \xi), \xi) ds \right) = |\det(\partial_x q_0(t; x, \xi))|.$$

Utilizing (3.14) and taking the limit $t \rightarrow +\infty$, we obtain Lemma 3.3. \square

4. SEMICLASSICAL APPROXIMATION OF TIME-DELAY

Our study of time-delay operator $T(h)$ is based on (1.2). Let J be an interval of nontrapping energy. For $f \in C_0^\infty(J)$, one has

$$(4.1) \quad f(H_0)T(h) = f_1(H_0^h)S(h)^*[S(h), A(h)]$$

where $f_1(t) = f(t)/2t$, $t > 0$. Therefore, the semiclassical approximation of $T(h)$ is closely related to that for $S(h)$. However, as already remarked in [27], although $S(h)$ can only be approximated by Fourier integral operators, $T(h)$ can be well approximated by h - ψ DOs. We begin with constructing approximation for wave operators.

For $d > 0$, take $\chi \in C^\infty(\mathbf{R})$ such that $\chi(\lambda) = 0$ for $\lambda < d/3$ and $\chi(\lambda) = 1$ for $\lambda > d/2$. For $\varepsilon > 0$ small enough, choose $\rho \in C^\infty(\mathbf{R})$ such that $\rho(r) = 0$ for $r < -\varepsilon$ and $\rho(r) = 1$ for $r \geq 0$. Let $\theta \in C_0^\infty(\mathbf{R}^n)$, $\theta(x) = 1$ for $|x| \leq 1$. Then we have for any $R > 0$,

$$\chi(H_0^h) = \theta\left(\frac{x}{R}\right)\chi(H_0^h) + \widehat{b}_-(x, hD) + b_+(x, hD)$$

where

$$\widehat{b}_-(x, \xi) = (1 - \theta(\frac{x}{R}))\chi(|\xi|^2)(1 - \rho(\widehat{x} \cdot \widehat{\xi} - 1 + \varepsilon)),$$

$$b_+(x, \xi) = (1 - \theta(\frac{x}{R}))\chi(|\xi|^2)\rho(\widehat{x} \cdot \widehat{\xi} - 1 + \varepsilon).$$

Denote $E_d(h)$ the spectral projector of H_0^h onto the interval $[d, +\infty[$. From the estimates

$$\|\theta(\frac{x}{R})\chi(H_0^h)U_0(t, h)\langle x \rangle^{-1}\| \leq C\langle t \rangle^{-1}, \quad t \in \mathbb{R},$$

$$\|\tilde{b}_-(x, hD)U_0(t, h)\langle x \rangle^{-1}\| \leq C\langle t \rangle^{-1}, \quad t > 0,$$

we derive easily that for $f \in \text{Ran } E_d(h)$,

$$(4.2) \quad \begin{aligned} W_+(h)f &= \lim_{t \rightarrow +\infty} U(t, h)^* U_0(t, h)f \\ &= \lim_{t \rightarrow +\infty} U(t, h)^* b_+(x, hD)U_0(t, h)f \quad \text{in } L^2(\mathbb{R}^n). \end{aligned}$$

$$W_+(h) = \text{s-lim}_{t \rightarrow +\infty} \{U(t, h)^* J^+(a_+(h))U_0(t, h)[U_0(t, h)^* J^+(b)^* U_0(t, h)]\}$$

on $\text{Ran } E_d(h)$.

Lemma 4.1. *The limit $\text{s-lim}_{t \rightarrow +\infty} U_0(t, h)^* J^+(b)^* U_0(t, h)$ exists in $L^2(\mathbb{R}^n)$ and is equal to I on the range of $E_d(h)$:*

$$(4.3) \quad \lim_{t \rightarrow +\infty} U_0(t, h)^* J^+(b)^* U_0(t, h)f = f, \quad \forall f \in \text{Ran } E_d(h).$$

Proof. We give only the proof of (4.3). Notice that

$$J^+(b)^* u(x) = (2\pi h)^{-n} \iint_{\mathbb{R}^{2n}} e^{i(x-y) \cdot \xi / h} C(y, \xi; h) u(y) dy d\xi,$$

where $C(y, \xi; h) = e^{i(y \cdot \xi - \varphi_+(y, \xi)) / h} \bar{b}(y, \xi)$. For fixed $h > 0$, $C(h) \in S^0$. Therefore, if $C^W(x; hD; h)$ denotes the h - ψ DO with Weyl symbol $C(h)$, then

$$J^+(b)^* = C^W(x, hD; h) + r_1(x, hD; h)$$

where $r_1(h) \in S^{-1}$. An easy calculus gives

$$\text{s-lim}_{t \rightarrow +\infty} U_0(t, h)^* J^+(b)^* U_0(t, h) = \text{s-lim}_{t \rightarrow +\infty} U_0(t, h)^* C^W(x, hD)U_0(t, h).$$

$U_0(t, h)^* C^W(x, hD; h)U_0(t, h)$ is an h - ψ DO with the Weyl symbol $C(x + 2t\xi, \xi; h)$. As $t \rightarrow +\infty$, $b(x + 2t\xi, \xi) \rightarrow 1$ for $|\xi|^2 \geq d$ and by Lemma 3.1:

$$|(x + 2t\xi) \cdot \xi - \phi_+(x + 2t\xi; \xi)| \leq C\langle x + 2t\xi \rangle^{-\varepsilon_0} \rightarrow 0, \quad |\xi|^2 \geq d.$$

Therefore, $C^W(x + 2thD, hD; h)f \rightarrow f$ for any $f \in \text{Ran } E_d(h)$. This proves (4.3). \square

According to Lemma 4.1, for $f \in \text{Ran } E_d(h)$, (4.2) is true for any $\varepsilon > 0$, $R > 0$. Similarly,

$$(4.4) \quad W_-(h)f = \lim_{t \rightarrow -\infty} U(t, h)^* b_-(x, hD)U_0(t)f$$

where $b_-(x, \xi) = (1 - \theta(\frac{x}{R}))\chi(|\xi|^2)p(-1 + \varepsilon - \widehat{x} \cdot \widehat{\xi})$. We want to prove that for some parametrices $J^\pm(a_\pm(h))$,

$$(4.5) \quad W_\pm(h) = \text{s-lim}_{t \rightarrow \pm\infty} U(t, h)^* J^\pm(a_\pm(h))U_0(t, h) \quad \text{on } \text{Ran } E_d(h).$$

In fact J^\pm is just an outgoing (incoming) parametrix constructed as in [25]. However as remarked in [10], for phase functions given in [6], one cannot have (4.5). But we shall see that with the modification on the phase functions given in Lemma 3.1, (4.5) is really true.

Let $\varepsilon > 0$, $d > 0$ be sufficiently small and ϕ_\pm solutions to (3.7). For a symbol a , denote by $J^\pm(a)$ the Fourier integral operators defined by:

$$(4.6) \quad J^\pm(a)u(x) = (2\pi h)^{-n} \iint_{\mathbf{R}^{2n}} e^{i(\varphi_\pm(x, \xi) - y \cdot \xi)/h} a(x, \xi) u(y) dy d\xi$$

for $u \in \mathcal{S}(\mathbf{R}^n)$. Let $a_\pm(h)$ denote the h -parametrix constructed from ϕ_\pm given above, by a procedure completely similar to that described in [25] (see also [6]), such that if we define

$$(4.7) \quad r_\pm(h) = e^{-i\varphi_\pm/h} (H^h - |\xi|^2) (e^{i\varphi_\pm/h} a_\pm(h)),$$

then $r_\pm(h) \in S_{\pm}^{-1,1}(\varepsilon, d, R_1)$ and

$$(4.8) \quad r_\pm(x, \xi; h) = O(h^\infty \langle x \rangle^{-\infty}), \quad \text{for } (x, \xi) \in \Omega_\pm(2\varepsilon, 2d, 2R_1).$$

For b_+ defined in (4.2) with $R \gg R_1$, we can find $b \in S_+^0$ such that

$$(4.9) \quad \begin{cases} b(x, \xi) = 1 + O(\langle x \rangle^{-1}) & \text{in } \Omega_+(2 - \frac{\varepsilon}{2}, d, 4R), \\ J^+(a_+(h))J^+(b)^* = b_+(x, hD) \neq r_0(x, hD; h) \end{cases}$$

with $r_0(h) \in S_+^{-1,1}$. From (4.2) we have

$$W_+(h)f = \lim_{t \rightarrow +\infty} U(t, h)^* J^+(a_+(h)) U_0(t, h) f \quad \text{in } L^2(\mathbf{R}^n)$$

or formally,

$$(4.10) \quad W_+(h) = J^+(a_+(h)) + ih^{-1} \int_0^{+\infty} U(t, h)^* J^+(r_+(h)) U_0(t, h) dt$$

on $\text{Ran } E_d(h)$. (4.10) shows that $J^+(a_+(h))$ is a good approximation for $W_+(h)$. Similarly, we can show that $J^-(a_-(h))$ is a good approximation for $W_-(h)$. Put

$$W_\pm(t, h) = U(t, h)^* J^\pm(a_\pm(h)) U_0(t, h).$$

Then by the completeness of wave operators, one has

$$(4.11) \quad S(h) = \text{s-lim}_{t \rightarrow +\infty} W_+(t, h)^* W_-(-t, h) \quad \text{on } \text{Ran } E_d(h).$$

Lemma 4.2. *Let J be an interval of nontrapping energy, $\chi \in C_0^\infty(J)$, $\phi \in C_0^\infty(\mathbf{R}^+)$. Then for any $s \in [-1, 1]$, one has*

$$(4.12) \quad \|\langle A \rangle^s \chi(H^h) W_\pm(t, h) f(H_0^h) \langle A \rangle^{-s}\| \leq C$$

uniformly in $t \in T$, $h \in]0, h_0]$. Here $A = h(x \cdot D_x + D_x \cdot x)/2$, $\langle A \rangle = (1 + |A|^2)^{1/2}$.

Proof. Consider only $W_+(t, h)$. By the construction of $J^+(a_+(h))$, one has

$$(4.13) \quad W_+(t, h) = J^+(a_+(h)) + ih^{-1} \int_0^t U(-\tau, h) J^+ U_0(\tau, h) d\tau$$

where $J^+ = J^+(R_+(h))$. Clearly it suffices to discuss the commutator of the second term in the right-hand side of (4.13) with A . Take $\chi_1, \chi_2 \in C_0^\alpha(J)$ with $\chi_1 = 1$ on $\text{supp } \chi$ and $\chi_2 = 1$ on $\text{supp } \chi_1$. We can write

$$\begin{aligned} & [A, \chi(H^h)U(\tau, h)^* J^+ U_0(\tau, h)f(H_0^h)] \\ &= [A, \chi(H^h)]U(\tau, h)^* \chi_1(H^h)J^+ U_0(\tau, h)f(H_0^h) \\ & \quad + \chi(H^h)U(\tau, h)^* [A, J^+]U_0(\tau, h)f(H_0^h) \\ & \quad + \chi(H^h)U(\tau, h)[A, \chi_1(H^h)]J^+ U_0(\tau, h)f(H_0^h) \\ & \quad + \chi(H^h)U(\tau, h)^* J^+ U_0(\tau, h)[A, f(H_0^h)] \\ & \quad + \int_0^\tau \chi(H^h)U(s, h)^* \tilde{V}U(s - \tau, h)ds \chi_1(H^h)J^+ U_0(\tau, h)f(H_0^h) \\ & \quad + 2\tau \chi(H^h)U(\tau, h)^* (J^+ H_0^h - H^h J^+)U_0(\tau, h)f(H_0^h) \\ & \equiv \sum_{j=1}^6 B_j(\tau, h). \end{aligned}$$

Here we used the commutator relation

$$[A, U(t, h)] = 2tH^h U(t, h) + \int_0^t U(t-s, h)\tilde{V}U(s)ds$$

with $\tilde{V} = x \cdot \nabla_x V - 2V$. To estimate the $B_j(t, h)$ we need the following two estimates:

$$(4.14) \quad \left\| \int_0^\tau \chi(H^h)U(s, H)^* \tilde{V}U(s; h)\chi_1(H^h)ds \right\| \leq C$$

uniformly in $\tau \in \mathbf{R}$, $h \in]0, h_0]$, and for any $\varphi \in C_0^\infty(J)$, $b_1 \in S_+^{-1/2} \cap S_-^{-1/2}$,

$$(4.15) \quad \left| \int_0^t \|b_1(x, hD)U(s, h)\varphi(H^h)g\|^2 ds \right| \leq C\|g\|^2, \quad g \in L^2(\mathbf{R}^n)$$

uniformly in $t \in \mathbf{R}$ in $h \in]0, h_0]$. (4.14) was proved in [27]. To prove (4.15) recall that I proved in [25] that for $b_\pm \in S_\pm^0$, $S > \frac{1}{2}$,

$$\|\langle x \rangle^{-s} R(\lambda \pm i0, h)b_\pm(x, hD)\langle x \rangle^{s-1}\| \leq Ch^{-1}, \quad \lambda \in J.$$

Therefore if $b \in S_+^0 \cap S_-^0$, and $s = 0, 1$, one has

$$\|\langle x \rangle^{-s} b(x, hD)R(\lambda \pm i0; h)b(x, hD)\langle x \rangle^{s-1}\| \leq Ch^{-1}.$$

By an easy interpolation, one obtains

$$(4.16) \quad \|\langle x \rangle^{-1/2} b(x, hD) R(\lambda \pm i0, h) b(x, hD) \langle x \rangle^{-1/2}\| \leq Ch^{-1}.$$

From (4.16), one can derive (4.15) by the local H^h -smoothness of $b_1(x, hD)$. See also the proof of Proposition 2.3 in [27]. Since $r_+(h) \in S_+^{-1,1}(\varepsilon, d, R_1)$ and $r_+(x, \xi; h) = O(h^\infty \langle x \rangle^{-\infty})$ in $\Omega_+(2\varepsilon, 2d, 2R_1)$, we can write

$$J^+ = h\{b_1(x, hD)J^+(r_1(h))b_2(x, hD) + J^+(r_2(h))\}$$

where $b_j \in S_+^{-1/2} \cap S_-^{-1/2}$, $r_1(h) \in S^{0,0}$ and $r_2(h) \in S^{-\infty,0}$. From (4.15) and a similar result for $U_0(t, h)$, we derive easily that

$$(4.17) \quad \left\| \int_0^t U(\tau, h)^* \varphi(H^h) J^+ U_0(\tau, h) f(H_0^h) d\tau \right\| \leq Ch$$

uniformly in $t \in \mathbf{R}$ and $h \in]0, h_0]$. Since $[A, J^+]$ has similar properties as J^+ , it follows from (4.17) that

$$(4.18) \quad \left\| \int_0^t h^{-1} B_j(\tau, h) d\tau \right\| \leq C, \quad j = 1, 2,$$

uniformly in t and h . By the relations (see [27])

$$\begin{aligned} i[A, \chi_2(H^h)] &= h(2H^h \chi_2'(H^h) + R_1(h), \\ i[A, f(H_0^h)] &= h2H_0^h f'(H_0^h), \end{aligned}$$

where $R_1(h)$ is an h -admissible operator in the sense of [4] with order $-1 - \varepsilon_0$ in x , we can easily prove that (4.18) is also true for $j = 3, 4$. For $j = 5$, one has

$$(4.19) \quad \begin{aligned} \left\| \int_0^t B_5(\tau, h) d\tau \right\| &\leq \left(\sup_{\tau} \left\| \int_0^{\tau} \chi(H^h) U(s, h)^* \tilde{V} \chi_1(H^h) U(s, h) ds \right\| \right) \\ &\quad \times \left\| \int_0^t \chi_2(H^h) U(\tau, h)^* J^+ U_0(\tau, h) f(H_0^h) d\tau \right\| \\ &\leq Ch \end{aligned}$$

uniformly in t . Here one used (4.14) and (4.17). Finally for $j = 6$, we notice that ϕ_+ solves the eikonal equation on $\text{supp } r_+(h)$. Hence,

$$J^+ H_0^h - H^h J^+ = J^+(\tilde{r}(h))$$

where $\tilde{r}(h) \in S_+^{-2,2}(\varepsilon, d, R_1)$ and

$$\tilde{r}(x, \xi; h) = O(h^\infty \langle x \rangle^{-\infty}) \quad \text{on } \Omega_+(2\varepsilon, 2d, 2R_1).$$

Since $\tau f(H_0^h) U_0(\tau, h)$ can be written as

$$\tau f(H_0^h) U_0(\tau, h) = i f_1(H_0^h) [A, U_0(\tau, h)] f_2(H_0^h),$$

where $f_1(\lambda) = \frac{1}{2\lambda}f(\lambda)$ and $f_2 \in C_0^\infty(\mathbf{R}^+)$, $f_2 = 1$ on $\text{supp } f$, we can derive from (4.17) that

$$(4.20) \quad \left\| \int_0^t B_6(\tau, h) \langle A \rangle^{-1} d\tau \right\| \leq Ch, \quad h \in]0, h_0],$$

uniformly in t . Now from (4.14) and (4.18) to (4.20), it follows that

$$\| \langle A \rangle \chi(H^h) W_+(t, h) f(H_0^h) \langle A \rangle^{-1} \| \leq C$$

uniformly in t and h . Similarly we can prove that

$$\| \langle A \rangle^{-1} \chi(H^h) W_+(t, h) f(H^h) \langle A \rangle \| \leq C'$$

uniformly in t and h . Equation (4.12) follows by interpolation. \square

Now let $\chi \in C_0^\infty(J)$. Take $\chi_j \in C_0^\infty(J)$, $j = 1, 2$, such that $\chi_1 = 1$ on $\text{supp } \chi$ and $\chi_2 = 1$ on $\text{supp } \chi_1$. Put

$$S(t, h) = \chi_2(H_0^h) W_+(t, h)^* \chi_1(H^h) W_-(-t, h) \chi(H_0^h).$$

By (4.11) and Lemma 4.2,

$$(4.21) \quad \chi_2(H_0^h) S^*(h) A S(h) \chi(H_0^h) f = \lim_{t \rightarrow +\infty} S(t, h)^* A S(t, h) f$$

for all $f \in \mathcal{D}(A)$. We can write

$$(4.22) \quad \begin{aligned} S(t, h)^* A S(t, h) &= S(t_0, h)^* A S(t_0, h) \\ &+ \int_{t_0}^t \{ R_1(r)^* A S(r, h) + S(r, h)^* A R_1(r) \\ &+ R_2(r)^* A S(r, h) + S(r, h)^* A R_2(r) \} dr \end{aligned}$$

where

$$\begin{aligned} R_1(t) &= \chi_2(H_0^h) R_+(t, h)^* \chi_1(H^h) W_-(-t, h) \chi(H_0^h), \\ R_2(t) &= -\chi_2(H_0^h) W_+(t, h)^* \chi_1(H^h) R_-(-t, h) \chi(H_0^h), \quad \text{and} \\ R_\pm(t) &= ih^{-1} U(t, h)^* J^\pm(r_\pm(h)) U_0(t, h). \end{aligned}$$

Recall that $r_\pm(h) = e^{-i\phi_\pm/h}(H^h - |\xi|^2)(e^{-i\phi_\pm/h}a_\pm(h))$ and $a_+(h)$ ($a_-(h)$, resp.) is an outgoing (incoming, resp.) h -parametrix.

Proposition 4.3. *Let $b \in S_+^0 \cap S_-^0$. Assume that $\text{supp } b \subset \{(x, \xi) \in \mathbf{R}^{2n}; |\widehat{x} \cdot \widehat{\xi}| \leq 1 - \sigma\}$, $\sigma > 0$. Let $J =]a, b[$ be an interval of nontrapping energy. For $0 < \varepsilon_1 \ll \sigma$, $0 < d_1 \ll a$, let $a_\pm(h)$ be the parametrix constructed in $\Omega_\pm(\varepsilon_1, d_1, R_1)$ as before. Then with the above notations, there exists $T_0 > 0$ such that for any $N \in \mathbf{N}$,*

$$(4.23) \quad \|AR_j(t, h)b(x, hD)\| \leq C_N h^N \langle t \rangle^{-N}, \quad \text{for } t > T_0, \quad j = 1, 2.$$

Proof. Consider first the case $j = 2$. By Lemma 4.2,

$$\begin{aligned} & \|AR_2(t, h)b(x, hD)\| \\ & \leq Ch^{-1}\|A\chi_1(H^h)U(t, h)J^-(r_-(h))U_0(-t, h)\chi(H_0^h)b(x, hD)\| \\ & \leq Ch^{-1}\langle t\rangle\|J^-(\tilde{r}_-(h))U_0(-t, h)\chi(H_0^h)b(x, hD)\| \end{aligned}$$

where $\tilde{r}_-(h)$ is determined by $AJ^-(r_-(h)) = J^-(\tilde{r}_-(h))$. Consequently, $\tilde{r}_-(h) \in S_+^{0,1}(\varepsilon_1, d_1, R_1)$ and $\tilde{r}_-(\chi, \xi; h) = O(h^\infty\langle x\rangle^{-\infty})$ in $\Omega_+(2\varepsilon_1, 2d_1, 2R_1)$. $\tilde{r}_-(h)$ can be decomposed as

$$\tilde{r}_-(h) = r_1(h) + r_2(h) + r_3(h)$$

where $\text{supp } r_1(h) \subset \{(x, \xi); |x| \leq \frac{7}{2}R_1\}$, $\text{supp } r_2(h) \subset \{(x, \xi), -1 + \varepsilon_1 < \hat{x}, \hat{\xi} < -1 + 3\varepsilon_1, |\xi|^2 \geq 3d_1, |x| \geq 3R_1\}$ and $r_3(h) \in S^{-\infty, +\infty}$. Since $3\varepsilon_1 < \sigma$, we derive from Corollary 2.2 (for $U_0(t, h)$) that there exists $T_0 > 0$ such that

$$\|J^-(\tilde{r}_-(h))U_0(-t, h)\chi(H_0^h)b(x, hD)\| = O(h^\infty\langle t\rangle^{-\infty}), \quad t > T_0.$$

This proves (4.23) for $j = 2$.

For $j = 1$, one can write

$$(4.24) \quad W_-(-t, h) = J^-(a_-(h)) - ih^{-1} \int_0^t U(s; h)J^-(r_-(h))U_0(-s, h)ds.$$

By Corollary 2.2 and the construction of $r_\pm(h)$, we can easily check that there is $T_0 > 0$ such that for any $m \geq 0$,

$$(4.25) \quad \|AJ^+(r_+(h))^*U(t+s; h)\chi_1(H^h)J^-(r_-(h))\langle x\rangle^m\| = O(h^\infty\langle t+s\rangle^{-\infty})$$

for $t+s > T_0$. From (4.25), it follows that

$$\begin{aligned} (4.26) \quad & \left\| AJ^+(r_+(h))^* \int_0^t U(t+s; h)\chi_i(H^h)J^-(r_-(h))U_0(-s, h)\chi(H_0^h)b(x, hD)ds \right\| \\ & = O(h^\infty\langle t\rangle^{-\infty}) \end{aligned}$$

for $t > T_0$. Since $b \in S_+^0(\sigma, d_2, R_2)$, choosing a larger $R_1 = R_1(\varepsilon_1, d_1)$ if necessary (see (3.8)), we can prove by the calculus of Fourier integral operators that

$$J^-(a_-(h))b(x, hD) = \tilde{b}(x, hD; h)J^-(\tilde{a}(h)) + J^-(\tilde{r}(h))$$

where $\tilde{r}(h) \in S^{-\infty, \infty}$, $\tilde{a}(h) \in S^{0,0}$, and $\tilde{b}(h) \in S_+^{0,0} \cap S_-^{0,0}$ with support contained in $\{|\hat{x}, \hat{\xi}| < 1 - \sigma/2\}$. Therefore by the construction of $r_+(h)$, there exists some $T_0 > 0$ such that

$$\begin{aligned} (4.27) \quad & \|AJ^+(r_+(h))^*U(t, h)\chi_1(H^h)J^-(a_-(h))\chi(H_0^h)b(x, hD)\| \\ & \leq O(h^\infty\langle t\rangle^{-\infty}) + C\|AJ^+(r_+(h))^*U(t, h)\chi_1(H^h)b(x, hD; h)\| \\ & \leq O(h^\infty\langle t\rangle^{-\infty}), \quad t > T_0. \end{aligned}$$

From (4.24), (4.26), and (4.27), it results that:

$$\begin{aligned} & \|AR_1(t)b(x, hD)\| \\ & \leq C\langle t\|AJ^+(r_+(h))^*U(t, h)\chi_1(H^h)W_-(t, h)\chi_2(H_0^h)b(x, hD)\| \\ & \leq O(h^\infty\langle t\rangle^{-\infty}) \end{aligned}$$

for $t > T_0$. This finishes the proof of Proposition 4.3. \square

Now since $T(h)f(H_0^h) = f_1(H_0^h)\{A - S(h)^*AS(h)\}\chi(H_0^h)$, where $f_1(\lambda) = \frac{1}{2\lambda}f(\lambda)$ and $\chi(\lambda) = 1$ for $\lambda \in \text{supp } f$, it follows from (4.21) and (4.22) and Proposition 4.3 that the following result holds.

Corollary 4.4. *Let J be an interval of nontrapping energy, $f \in C_0^\infty(J)$. Assume $a, b \in S_+^0 \cap S_-^0$. Then there exists some $T_0 > 0$ such that*

$$(4.28) \quad \|a(x, hD)\{T(h)f(H_0^h) - f_1(H_0^h) \times (A\chi(H_0^h) - S(t, h))^*AS(t, h)\}b(x, hD)\| = O(h^\infty)$$

uniformly in $t > T_0$. Here $S(t, h)$ is defined as in (4.21). In particular, $a(x, hD)T(h)f(H_0^h)b(x, hD)$ is an h -pseudodifferential operator.

Proof. By the calculus of Fourier integral operators (see, e.g., [25, Appendix]) and the semiclassical Egorov theorem (see Theorem A, Appendix), one sees clearly that $S(t, h)^*AS(t, h)$ is an h -pseudodifferential operator. Equation (4.28) follows from Proposition 4.3. From this it is also clear that the remainder in equation (4.28) is in fact a smoothing operator. So it follows that $a(x, hD)T(h)f(H_0^h)b(x, hD)$ is an h -pseudodifferential operator with symbols equal to that of $a(x, hD)f_1(H_0^h)(A\chi(H_0^h) - S(t, h)^*AS(t, h))b(x, hD)$ up to an error of order $O(h^\infty)$. \square

In the following we shall compute the h -principal symbol of

$$a(x, hD)f(H_0^h)T(h)b(x, hD).$$

Our computation is based on the results of Lemmas 3.2 and 3.3.

Proposition 4.5. *For $b \in S^m$, $a_j \in S_+^0 \cap S_-^0$, there exists $t_0 > 0$ such that for $t > t_0$,*

$$a_1(x, hD)W_-(-t, h)^*b(x_1hD)W_-(-t, h)a_2(x, hD) = C(x, hD; h).$$

Here $C(h) \in S^{m,0}$ admits an asymptotic expansion: $C(h) \sim \sum_{j=0}^\infty h^j C_j$ with C_0 given by

$$C_0(x, \xi) = a_1(x, \xi)a_2(s, \xi)b(\Omega_-^{\text{cl}}(x, \xi)).$$

Ω_-^{cl} is the classical incoming wave operator.

Proof. Recall first that if $d_j \in S^{mj}$, $j = 1, 2$, then

$$J^-(d_1)^*J^-(d_2) = e(x, hD; h)$$

is an h -pseudodifferential operator with symbol $e(h) \in S^{m_1+m_2,0}$. In addition, $e(h) \sim \sum_{j=0}^{\infty} h^j e_j$ and e_0 is given by

$$(4.29) \quad e_0(x, \xi) = d_1(y_-(x, \xi), \xi) d_2(y_-(x, \xi), \xi) \left| \frac{\partial y_-(x, \xi)}{\partial x} \right|.$$

By the Egorov theorem, $U(t, h)^* b(x, hD) U(t, h)$ is an h -pseudodifferential operator with h -principal symbol $b(\phi^t(x, \xi))$, where $\phi^t = (q(t), p(t))$ is the Hamiltonian flow. From (4.29) one sees that

$$a_1(x, hD) W_-(-t, h)^* b(x, hD) W_-(-t, h) a_2(x, hD)$$

is an h -pseudodifferential operator with h -principal symbol C_0 :

$$C_0(x, \xi) = a_1(x, \xi) a_2(x, \xi) d(x - 2t\xi, \xi),$$

where

$$d(x, \xi) = a_{0,-}(y_-(x, \xi), \xi)^2 \left| \frac{\partial y_-}{\partial x}(x, \xi) \right| d_1(x, \xi),$$

$$d_1(x, \xi) = b(\phi^t(y_-(x, \xi)), \nabla_x \phi_-(y_-(x, \xi), \xi)).$$

For $(x, \xi) \in \text{supp } a_1$, one has

$$|x - 2t\xi| \geq \delta(|x| + 2t|\xi|), \quad \delta > 0, \quad \forall t > 0.$$

Since $y_-(x, \xi) - x = O(\langle x \rangle^{-\varepsilon_0})$, there exists some $t_0 > 0$ such that $(x, \xi) \in \text{supp } a_1$ implies that $(y_-(x - 2t\xi, \xi), \xi) \in \Omega_-(2\varepsilon_1, 2d_1, 2R_1)$ for $t > t_0$. According to Lemma 3.3, for $(y_-, \xi) \in \Omega_-(2\varepsilon_1, 2d_1, 2R_1)$, $a_0(y_-, \xi)^2 |\partial y_- / \partial x| = |\det(\partial_x \partial_\xi \phi_-(y_-, \xi))| = 1$, since $\partial_\xi \phi_-(y_-(x, \xi), \xi) = x$. Therefore for $t > t_0$, $C_0(x, \xi) = a_1(x, \xi) a_2(x, \xi) d_1(x - 2t\xi, \xi)$. Put $\phi_0^t(x, \xi) = (x + 2t\xi, \xi)$. From the definition of $\Omega_-^{\text{cl}}(x, \xi)$ [16], one sees that

$$\Omega_-^{\text{cl}}(x, \xi) = (q_-(0; x, \xi), p_-(0; x, \xi)),$$

where $(q_-(t), p_-(t))$ is the solution of the Hamilton system (1.3) with the condition

$$\begin{cases} q_-(t; x, \xi) = x + 2t\xi + o(1), & t \rightarrow -\infty, \\ p_-(t; x, \xi) = \xi + o(1), & t \rightarrow -\infty. \end{cases}$$

From the proof of Lemma 3.2, it is clear that for $(x, \xi) \in \Omega_-(2\varepsilon_1, 2d_1, 2R_1)$, $R_1 > 0$ large enough, $q_-(t; x, \xi) = q(t; y_-(x, \xi), \nabla_x \phi_-(y_-(x, \xi), \xi))$. This means

$$\Omega_-^{\text{cl}}(x, \xi) = (y_-(x, \xi), \nabla_x \phi_-(y_-(x, \xi), \xi)) \quad \text{for } (x, \xi) \in \Omega_-(2\varepsilon_1, 2d_1, 2R_1).$$

But for $(x, \xi) \in \text{supp } a_1$, $t > t_0$, one always has

$$\phi_0^{-t}(x, \xi) \in \Omega_-(2\varepsilon_1, 2d_1, 2R_1).$$

Therefore,

$$d_1(\phi_0^{-t}(x, \xi)) = b(\phi^t \circ \Omega_-^{\text{cl}} \circ \phi_0^{-t}(x, \xi)) = b(\Omega_-^{\text{cl}}(x, \xi)).$$

In the last equality, we have used the intertwining property of classical wave operators. \square

Proposition 4.6. *With the notation of Corollary 4.4, for any $a, b \in S_+^0 \cap S_-^0$, there is $t_0 > 0$ such that $a(x, hD)f_1(H_0^h)(A - S(t, h)^*AS(t, h))b(x, hD)$ is an h -pseudodifferential operator of order zero in x and that its symbol has an expansion of the form: $C(h) \sim \sum C_j h^j$, where C_j is independent of $t > t_0$ for all $j \geq 0$ and*

$$C_0(x, \xi) = f(\xi^2)a(x, \xi)b(x, \xi)\tau \circ \Omega_-^{\text{cl}}(x, \xi).$$

Here τ is the classical time-delay function.

Proof. Recall first that $\chi_1(H^h)$ is an h -pseudodifferential operator with h -principal symbol $\chi_1(|\xi|^2 + V(x))$. From Proposition 4.5, it follows immediately that $a(x, hD)f_1(H^h)S(t, h)^*AS(t, h)b(x, hD)$ is an h -pseudodifferential operator and when $t > t_0$, its h -principal symbol is given by

$$d(x, \xi) = a(x, \xi)b(x, \xi)f_1(|\xi|^2)F(\Omega_-^{\text{cl}}(x, \xi)),$$

where F is the h -principal symbol of $W_+(t, h)\chi_2(H_0^h)A\chi_2(H_0^h)W_+(t, h)^*$;

$$(4.30) \quad F(x, \xi) = G(\phi^t(x, \xi))$$

and

$$G(x, \xi) = a_{0,+}(x, \eta_+(x, \xi))^2 \left| \frac{\partial \eta_+}{\partial \xi} \right| \{ \nabla_\xi \phi_+(x, \eta_+(x, \xi)) \cdot \eta_+(x, \xi) - 2t|\eta_+(x, \xi)|^2 \chi_2(|\eta_+(x, \xi)|^2)^2 \}.$$

By Lemma 3.2, $\eta_+(x, \xi) = \xi - \int_0^{+\infty} \nabla V(q(t; x, \xi)) dt$ is invariant by ϕ^t ;

$$\eta_+(\phi^t(x, \xi)) = \eta_+(x, \xi), \quad \forall t \in \mathbb{R}, \quad (x, \xi) \in \Omega_+(2\varepsilon_1, 2d_1, 2R_1).$$

According to Lemma 3.3, for $(x, \xi) \in \Omega_+(2\varepsilon_1, 2d_1, 2R_1)$, one has

$$a_{0,+}(x, \eta_+(x, \xi))^2 \left| \frac{\partial \eta_+}{\partial \xi} \right| = 1.$$

By a direct computation, we can check that

$$\nabla_\xi \phi_+(q(t; x, \xi), \eta_+(x, \xi)) = \nabla_\xi \phi_+(x, \eta_+(x, \xi)) + 2t\eta_+(x, \xi),$$

for $(q(t), \xi) \in \Omega_+(2\varepsilon_1, 2d_1, 2R_1)$ with possibly larger R_1 . On the other hand, for $(x, \xi) \in \text{supp } a$, one has $\widehat{x} \cdot \widehat{\xi} > -1 + \sigma$, $\sigma \gg \varepsilon_1 > 0$. If $|x| \geq R_2$ with R_2 large enough, then by the expression for Ω_-^{cl} :

$$\Omega_-^{\text{cl}}(x, \xi) = (y_-(x, \xi), \nabla_x \phi_-(y_-(x, \xi), \xi)),$$

one can derive that

$$\phi^t(\Omega_-^{\text{cl}}(x, \xi)) \in \Omega_+(2\varepsilon_1, 2d_1, 2R_1) \quad \text{for all } t > 0.$$

If $|x| < R_2$, $|\xi|^2 \in \text{supp } f_1$, then $|\Omega_-^{\text{cl}}(x, \xi)| \leq C$, where C depends only on R_2 and f_1 . By the nontrapping assumption, there exists $t_0 > 0$ such that

$$\phi^t(\Omega_-^{\text{cl}}(x, \xi)) \in \Omega_+(2\varepsilon_1, 2d_1, 2R_1), \quad t > t_0.$$

Summing up, we have proved that there exists $t_0 > 0$ large enough, such that

$$\phi^t(\Omega_-^{\text{cl}}(x, \xi)) \in \Omega_+(2\varepsilon_1, 2d_1, 2R_1)$$

for $t > t_0$ and $(x, \xi) \in \text{supp } a$, $|\xi|^2 \in \text{supp } f_1$. From the definition of τ and Lemma 3.2, one see that

$$\tau(x, \xi) = \frac{1}{2p(x, \xi)}(\tau_-(x, \xi) - \tau_+(x, \xi)),$$

where

$$(A) \quad \tau_{\pm}(x, \xi) = \nabla_{\xi} \phi_{\pm}(x, \eta_{\pm}(x, \xi)) \cdot \eta_{\pm}(x, \xi), \quad (x, \xi) \in \Omega_{\pm}(2\varepsilon_1, 2d_1, 2R_1).$$

Now from (4.30) and (A), it follows that

$$(4.31) \quad d(x, \xi) = a(x, \xi)b(x, \xi)f_1(|\xi|^2)\tau_+ \cdot \Omega_-^{\text{cl}}(x, \xi).$$

Here we have used the relation $|\eta_+(\Omega_-^{\text{cl}}(x, \xi))|^2 = |\xi|^2$ and $\chi_2 = 1$ on $\text{supp } f_1$. Remark that τ_{\pm} is well defined on $p^{-1}(J)$, $p(x, \xi) = |\xi|^2 + V(x)$. For $|\xi|^2 \in J$, we can check directly that $\tau_- \cdot \Omega_-^{\text{cl}}(x, \xi) = x \cdot \xi$. From (A) and (4.31), it results that for $t > t_0$, the h -principal symbol of

$$a(x, hD)f_1(H_0^h)(A - S(t, h)^*AS(t, h))^*b(x, hD)$$

is $f(|\xi|^2)a(x, \xi)b(x, \xi)\tau \circ \Omega_-^{\text{cl}}(x, \xi)$. \square

Now we are able to give easily the proof of Theorems 1 and 2.

Proof of Theorem 1. If $a, b \in S_+^0 \cap S_-^0$, the results follow from (4.1), Corollary 4.4, and Proposition 4.6. If $a, b \in C_0^\infty(R_x^n)$, notice that by Corollary 2.2 for any $N \geq 0$, one has

$$(4.32) \quad \|A(h)R_j(t, h)g(x)\| \leq C_N h^N \langle t \rangle^{-N}, \quad t > t_0, \quad j = 1, 2.$$

Here $R_j(t, h)$ is defined in Proposition 4.3 and $g = a$ or b . From (4.22) it follows that $\|a(x)\{T(h)f(H_0^h) - f_1(H_0^h)(A - S(t, h)^*AS(t, h))\}b(x)\| = O(h^\infty)$ uniformly in $t > t_0$. Now the desired results can be derived as in Proposition 4.6. The details are omitted. \square

Proof of Theorem 2. Recall that $\chi(\overline{H}_0^h)\overline{T}(h)$ is uniformly bounded in $\mathcal{L}(L^2)$ by the nontrapping assumption on J , (see [27]):

$$\|\chi(\overline{H}_0^h)\overline{T}(h)\| \leq C < +\infty, \quad h \in]0, h_0].$$

By an argument of density, it suffices to prove (1.9) for $f, g \in C_0^\infty(R^n)$. Now take $a \in C_0^\infty(R^n)$, $a(x) = 1$ for x near x_0 . Put

$$f_h = U_h W_h(x_0, \xi_0)f, \quad g_h = U_h W_h(x_0, \xi_0)g.$$

Then for $h > 0$ small enough, one has

$$\langle W_h(x_0, \xi_0)^* \chi(\bar{H}_0^h) \bar{T}(h) W_h(x_0, \xi_0) f, g \rangle = \langle aT(h) \chi(H_0) a f_h, g_h \rangle.$$

Now (1.9) follows from Theorem 1 by the arguments already used in [27]. See the proof of Theorem 6.3 in [27] and Theorem B in Appendix. \square

APPENDIX. A SEMICLASSICAL EGOROV THEOREM

For the reader's convenience, we state here a semiclassical Egorov theorem used in this paper, which is a particular case of Theorem 3.7 in [22]. See also [17] when $b \in S^0$.

Theorem A. *Let $b \in S^m$, $m \in \mathbb{R}$. Then under the assumption (2.1) on V , $U(t, h)^* b(x, hD) U(t, h)$ is an h - ψ DO with symbol $f(t, h) \in S^{m,0}$. Moreover, $f(t, h) \sim \sum_{j=0}^{\infty} h^j f_j(t)$, where each $f_j(t)$ can be computed from b and the Hamilton flow ϕ^t of (1.3). In particular, one has $\text{supp } f_j(t) \subset \text{supp } b(\phi^t)$, for all $j \geq 0$, and $f_0(t) = b(\phi^t)$.*

From Theorem A we can easily derive the following result.

Theorem B. *Let $b \in S^m$ and U_h be defined as in Introduction. Then for any $(x_0, \xi_0) \in \mathbb{R}^{2n}$ and $f \in \mathcal{S}(\mathbb{R}^n)$, one has*

$$\begin{aligned} \lim_{h \rightarrow 0_+} W_h(x_0, \xi_0)^* U_h^* U(t, h)^* b(x, hD) U(t, h) U_h W_h(x_0, \xi_0) f \\ = b(q(t), p(t)) f, \quad \text{in } L^2(\mathbb{R}^n). \end{aligned}$$

Here $(q(t), p(t))$ is a solution to (1.3) with initial data (x_0, ξ_0) .

For the proof of Theorem B, one is referred to [22]. Notice that in [22] the results are proved for Weyl ψ DOs. But there is a well-known equivalence between different quantifications [17].

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